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Indecomposable invariants of quivers for dimension $(2,\ldots,2)$ and maximal paths, II

A.A. Lopatin
Omsk Branch of Institute of Mathematics, SBRAS,
Pevtsova street, 13,
Omsk 644099 Russia
artem_lopatin@yahoo.com
http://www.iitam.omsk.net.ru/~lopatin/

Abstract

An upper bound on degrees of elements of a minimal generating system for invariants of quivers of dimension $(2,\ldots,2)$ is established over a field of arbitrary characteristic and its precision is estimated. The proof is based on the reduction to the problem of description of maximal paths satisfying certain condition.

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1 Introduction

We work over an infinite field K of arbitrary characteristic $\mathrm{char}(K)$. All vector spaces, algebras, and modules are over K unless otherwise stated and all algebras are associative.

This paper is a completion of [11] and we use the same notations as in [11]. Let us recall some of them. A *quiver* $\mathcal{Q} = (\mathrm{ver}(\mathcal{Q}), \mathrm{arr}(\mathcal{Q}))$ is a finite oriented graph, where $\mathrm{ver}(\mathcal{Q})$ is the set of vertices and $\mathrm{arr}(\mathcal{Q})$ is the set of arrows. The notion of quiver was introduced by Gabriel in [5] as an effective mean for description of different problems of the linear algebra.

The head (the tail, respectively) of an arrow a is denoted by a' (a'', respectively). We say that $a = a_1\cdots a_s$ is a *path* in $\mathcal{Q}$ (where $a_1,\ldots,a_s \in \mathrm{arr}(\mathcal{Q})$), if $a_1' = a_2'',\ldots,a_{s-1}' = a_s''$; and a is a *closed* path in a vertex v, if a is a path and $a_1'' = a_s' = v$. The head of the path a is $a' = a_s'$ and the tail of a is $a'' = a_1''$. Denote $\mathrm{ver}(a) = \{a_1'', a_1', \ldots, a_s'\}$, $\mathrm{arr}(a) = \{a_1,\ldots,a_s\}$, and $\deg(a) = s$. Given a closed path a and $w \in \mathrm{ver}(\mathcal{Q})$, we set $\deg_w(a) = \#\{i \,|\, a_i' = w,\ 1 \le i \le s\}$. A closed path a is called *primitive* if $\deg_w(a) = 1$ for all $w \in \mathrm{ver}(a)$. Denote by $m(\mathcal{Q})$ the maximal degree of primitive closed paths in $\mathcal{Q}$. Closed paths $a_1,\ldots,a_s$ in $\mathcal{Q}$ are called *incident* if $a_1' = \cdots = a_s'$.

For a quiver $\mathcal{Q}$ and a *dimension vector* $\boldsymbol{n} = (\boldsymbol{n}_v \,|\, v \in \mathrm{ver}(\mathcal{Q}))$ denote by $I(\mathcal{Q},\boldsymbol{n})$ the *algebra of invariants* of representations of $\mathcal{Q}$. The algebra $I(\mathcal{Q},\boldsymbol{n})$ is embedded into the algebra of (commutative) polynomials $K[x_{ij}(a) \,|\, a \in \mathrm{arr}(\mathcal{Q}),\ 1 \le i \le \boldsymbol{n}_{a'},\ 1 \le j \le \boldsymbol{n}_{a''}]$. Denote by $X_a = (x_{ij}(a))$ the $n_{a'} \times n_{a''}$ *generic* matrix and by $\sigma_k(X)$ the k-th coefficient in the characteristic polynomial of

an $n \times n$ matrix X, i.e.,

$$\det(\lambda E - X) = \lambda^n - \sigma_1(X)\lambda^{n-1} + \cdots + (-1)^n \sigma_n(X).$$

Theorem 1.1. *(Donkin [4]) The K-algebra $I(\mathcal{Q}, \boldsymbol{n})$ is generated by $\sigma_k(X_{a_s} \cdots X_{a_1})$ for all closed paths $a = a_1 \cdots a_s$ in $\mathcal{Q}$ (where $a_1, \ldots, a_s \in \operatorname{arr}(\mathcal{Q})$) and $1 \leq k \leq n_{a'}$.*

Notice that $I(\mathcal{Q}, \boldsymbol{n})$ has a grading by degrees that is given by the formula: $\deg(\sigma_k(X_{a_s} \cdots X_{a_1})) = ks$.

Investigation of $I(\mathcal{Q}, \boldsymbol{n})$ was originated from the partial case of a quiver with one vertex. Sibirskii [16], Razmyslov [15] and Procesi [13] described generators and relations in the case of characteristic zero field. As about the case of arbitrary characteristic, the first step was performed by Donkin in [3], where he established generators. Relations between generators of $I(\mathcal{Q}, \boldsymbol{n})$ were established by Domokos [1] in characteristic zero case and by Zubkov [17] in arbitrary characteristic case. Theorem 1.1 was generalized to the case of action of arbitrary classical linear groups in [10] using approach from [9].

By the Hilbert–Nagata Theorem on invariants, $I(\mathcal{Q}, \boldsymbol{n})$ is a finitely generated graded algebra. But the mentioned generating system is not finite. So it gives rise to the problem to find out a minimal (by inclusion) homogeneous system of generators (m.h.s.g.). Let $D(\mathcal{Q}, \boldsymbol{n})$ be the least upper bound for the degrees of elements of a m.h.s.g. of $I(\mathcal{Q}, \boldsymbol{n})$. Note that taking elements from Theorem 1.1 of the degree less or equal to $D(\mathcal{Q}, \boldsymbol{n})$ we obtain the finite system of generators. A *decomposable* invariant is equal to a polynomial in elements of strictly lower degree. Obviously, $D(\mathcal{Q}, \boldsymbol{n})$ is equal to the highest degree of indecomposable invariants.

In [11] we established an upper bound on $D(\mathcal{Q}, \boldsymbol{n})$ for an arbitrary quiver $\mathcal{Q}$ and $\boldsymbol{n} = (2, 2, \ldots, 2)$. In this paper we improve essentially the mentioned upper bound and estimate its precision (see Theorem 1.2 and Remark 1.3). Note that for a quiver with one vertex and $\boldsymbol{n} = (2)$ a m.h.s.g. was found in [16], [14], [2]; in case $\boldsymbol{n} = (3)$ a m.h.s.g. was described in [7], [8] and a system of parameters for a quiver with three loops was found in [6]. A m.h.s.g. for the algebra of semi-invariants of a quiver of dimension $(2, \ldots, 2)$ was established in [12]. References to other results on generating systems for invariants are given, for example, in [11].

Without loss of generality we can assume that $\mathcal{Q}$ is a *strongly connected* quiver, i.e., there exists a closed path in $\mathcal{Q}$ that contains all vertices of $\mathcal{Q}$ (for the details, see Section 1 of [11]).

For positive integers n, d, m define $M(n, d, m)$ as follows:

1) if $\operatorname{char}(K) = 2$, then

$$M(n,d,m) = \left\{ \begin{array}{rl} 2m, & \text{if } d = n = m \\ 2m(d - n + \frac{1}{2}), & \text{if } d < n + 2\left[\frac{n-1}{m}\right] \text{ and } n > m \geq 2 \\ m(d-n-1) + 2n, & \text{otherwise} \end{array} \right. ;$$

2) if $\operatorname{char}(K) \neq 2$, then

$$M(n,d,m) = \left\{ \begin{array}{ll} 2n, & \text{if } n = m \text{ and } d \in \{n, n+1\} \\ 3n, & \text{otherwise} \end{array} \right. .$$

Here $[\alpha]$ stands for the greatest integer that does not exceed α.

Denote by $\mathcal{Q}(n,d,m)$ the set of all strongly connected quivers $\mathcal{Q}$ with $\#\,\mathrm{ver}(\mathcal{Q})=n$, $\#\,\mathrm{arr}(\mathcal{Q})=d$, and $m(\mathcal{Q})=m$. A criterion when $\mathcal{Q}(n,d,m)$ is not empty is given by Lemma 2.2. For short, we write $D(n,d,m)$ for $\max\{D(\mathcal{Q},(2,\ldots,2))\,|\,\mathcal{Q}\in\mathcal{Q}(n,d,m)\}$. Our main result is the following theorem.

Theorem 1.2. *For $\mathcal{Q}(n,d,m)\neq\emptyset$ we have $D(n,d,m)\le M(n,d,m)$. Moreover,*

1) if $\mathrm{char}(K)=2$, then
$$D(n,d,m)\ge M(n,d,m)-m.$$

2) if $\mathrm{char}(K)\neq 2$, $d\ge n+2\left[\frac{n-1}{m}\right]+m$ or $n=m$, then
$$D(n,d,m)=M(n,d,m).$$

As immediate corollary of this theorem we obtain that if $\mathcal{Q}\in\mathcal{Q}(n,d,m)$, then the algebra of invariants $I(\mathcal{Q},(\delta_1,\ldots,\delta_n))$ with $\delta_1,\ldots,\delta_n\le 2$ is generated by elements of degree at most $M(n,d,m)$.

Remark 1.3. *Let $\mathrm{char}(K)=2$. In [11] we gave the following upper bound: $D(n,d,m)\le md$ for $\mathcal{Q}(n,d,m)\neq\emptyset$. By Theorem 1.2, for $m>2$ the deviation of this upper bound is*
$$md-D(n,d,m)\to\infty \text{ as } n,d\to\infty, \tag{1}$$
where we assume that m is fixed and $n,d\to\infty$ in such a way that at each step $\mathcal{Q}(n,d,m)\neq\emptyset$. But the deviation of the upper bound from Theorem 1.2 is less or equal to the constant m, i.e.,
$$0\le M(n,d,m)-D(n,d,m)\le m.$$

As in [11], for a quiver $\mathcal{Q}$ introduce an equivalence $\equiv$ on the set of all closed paths extended with an additional symbol 0. For any paths a,b such that ab is a closed path and any incident closed paths $a_1,a_2,\ldots$ we define

1. $ab\equiv ba$;
2. $a_{\sigma(1)}\cdots a_{\sigma(t)}\equiv \mathrm{sgn}(\sigma)\,a_1\cdots a_t$, where $t\ge 2$ and $\sigma\in\mathcal{S}_t$;
3. $a_1^2a_2\equiv 0$;
4. if $\mathrm{char}(K)=2$, then $a_1^2\equiv 0$; if $\mathrm{char}(K)\neq 2$, then $a_1a_2a_3a_4\equiv 0$.

We write $M(\mathcal{Q})$ for the maximal degree of a closed path a in $\mathcal{Q}$ satisfying $a\not\equiv 0$. The following lemma is Lemma 1.2 of [11], which was proved using [17].

Lemma 1.4. *Let $a=a_1\cdots a_s$ be a closed path in $\mathcal{Q}$, where $a_1,\ldots,a_s\in\mathrm{arr}(\mathcal{Q})$. Then $\mathrm{tr}(X_{a_s}\cdots X_{a_1})\in I(\mathcal{Q},(2,2,\ldots,2))$ is decomposable if and only if $a\equiv 0$.*

Remark 1.5. *Let $a=a_1\cdots a_s$ be a closed path in $\mathcal{Q}$, where $a_1,\ldots,a_s\in\mathrm{arr}(\mathcal{Q})$. If $q=\det(X_{a_s}\cdots X_{a_1})\in I(\mathcal{Q},(2,\ldots,2))$ is indecomposable, then a is a primitive closed path and $\deg(a)\le m$. Thus, $\deg(q)\le M(n,d,m)$.*

Section 2 contains necessary definitions and results from [11]. If $\mathrm{char}(K) \neq 2$, then the upper bound on $M(\mathcal{Q})$ is calculated in Lemma 2.4; otherwise, we establish the upper bound on $M(\mathcal{Q})$ in Theorems 5.1 and 6.2. In Lemma 7.1 we estimate a precision of the given upper bound. Taking into account Lemma 1.4 and Remark 1.5 together with the fact that $I(\mathcal{Q},(2,2,\ldots,2))$ is generated by indecomposable invariants, we complete the proof of Theorem 1.2.

In Sections 3–6 we assume that $\mathrm{char}(K) = 2$. Sections 3, 4, and 5 are dedicated to the proof of Theorem 5.1, which consists of two steps.

At first, we introduce the set of multidegrees $\Omega_2(\mathcal{Q})$ with the property that if h is a closed path and $\mathrm{mdeg}(h) \in \Omega_2(\mathcal{Q})$, then $h \not\equiv 0$ (see Section 3 and Remark 3.2). Moreover, Lemma 2.1 implies that $\Omega_2(\mathcal{Q})$ is the maximal (by inclusion) set with the given property. In Theorem 3.9 of Section 3 we give some upper bound on $|\underline{\delta}|$ for $\underline{\delta} \in \Omega_2(\mathcal{Q})$. Note that there can be a closed path $h \not\equiv 0$ such that $\mathrm{mdeg}(h) \notin \Omega_2(\mathcal{Q})$ (see Example 3.3).

During the second step we extract some information from the fact that $h \not\equiv 0$ (see Lemma 4.5). Then we find out a closed subpath c in h such that for two arrows b_1, b_2 of c we have $\deg_{b_1}(h) = \deg_{b_2}(h) = 1$ and some additional properties are valid (see Lemma 5.2). The main idea of the proof of Theorem 5.1 is to substitute c with a loop in order to obtain a quiver $\mathcal{G}$ with $\#\,\mathrm{arr}(\mathcal{G}) < \#\,\mathrm{arr}(\mathcal{Q})$ and to use induction hypothesis. The main difficulty is that we can not claim that c is a primitive closed path, thus we can not say that $\deg(c) \leq m$. To estimate $\deg(c)$ we apply Lemma 5.5.

Section 6 contains the proof of Theorem 6.2. In Section 7 we consider some examples in order to prove Lemma 7.1.

2 Auxiliary results

2.1 Notations

For a path $a = a_1 \cdots a_s$ in a quiver $\mathcal{Q}$, where $a_1, \ldots, a_s \in \mathrm{arr}(\mathcal{Q})$, and $b \in \mathrm{arr}(\mathcal{Q})$, $v \in \mathrm{ver}(\mathcal{Q})$, we set

- $\deg_b(a) = \#\{i \,|\, a_i = b,\ 1 \leq i \leq s\}$;
- $\deg_v(a) = \max\{m_1, m_2\}$, where $m_1 = \#\{i \,|\, a_i' = v,\ 1 \leq i \leq s\}$ and $m_2 = \#\{i \,|\, a_i'' = v,\ 1 \leq i \leq s\}$;
- $\deg_v^o(a) = \#\{i \,|\, a_i' = v,\ 1 \leq i \leq s-1\}$.

Let $\underline{\delta} \in \mathbb{N}^{\#\,\mathrm{arr}(\mathcal{Q})}$, where $\mathbb{N}$ stands for non-negative integers. Then the path a is called $\underline{\delta}$*-double* if a is a primitive closed path and $\delta_{a_i} \geq 2$ for all i. The definition of *strongly connected components* of an arbitrary quiver $\mathcal{G}$ is well known (for example, see Section 1 of [11]). The following notions were defined in Section 5 of [11]:

- the *multidegree* $\mathrm{mdeg}(a)$ of a path a;
- the *empty path* 1_v in a vertex v;
- a *subpath* of a path a;
- *h-restriction* of $\mathcal{Q}$ to V, where $V \subset \mathrm{ver}(\mathcal{Q})$ and h is a path in $\mathcal{Q}$ (see also Example 5.1 of [11]).

Denote by $\mathrm{path}(\mathcal{Q})$ the set of all paths and empty paths in $\mathcal{Q}$. If we consider a path, then we assume that it is non-empty unless otherwise stated; if we write $a \in \mathrm{path}(\mathcal{Q})$, then we assume that a path a can be empty.

Dealing with equivalences we use the following conventions. If we write $a \equiv b$, then we assume that a and b are closed paths in $\mathcal{Q}$. If we write ab for paths a and b, then we assume that $a' = b''$. To explain how we apply formulas to prove some equivalence $a \equiv b$ we split the word a into parts using dots.

For closed paths a, b we write $a \sim b$ if $a = c_1c_2$ and $b = c_2c_1$ for some $c_1, c_2 \in \mathrm{path}(\mathcal{Q})$. For $\underline{\delta}, \underline{\theta} \in \mathbb{N}^l$ we set $\underline{\delta} \geq \underline{\theta}$ if and only if $\delta_i \geq \theta_i$ for all i and define $|\underline{\delta}| = \delta_1 + \cdots + \delta_l$.

Let $x_1, \ldots, x_s$ be all arrows in $\mathcal{Q}$ from u to v, where $u, v \in \mathrm{ver}(\mathcal{Q})$. Then denote by $\check{x}$ any arrow from $x_1, \ldots, x_s$, by $\{\check{x}\}$ the set $\{x_1, \ldots, x_s\}$, and say that $\check{x}$ is an arrow from u to v. Schematically, we depict arrows $x_1, \ldots, x_s$ as

$$\begin{tikzcd} u \arrow[r, bend left, "\check{x}"] & v \end{tikzcd}.$$

For a path a in $\mathcal{Q}$ denote $\deg_{\check{x}}(a) = \sum_{i=1}^{s} \deg_{x_i}(a)$. As an example, an expression $\check{x}a_1 \cdots \check{x}a_k$ stands for a path $x_{i_1}a_1 \cdots x_{i_k}a_k$ for some $1 \leq i_j \leq s$ $(1 \leq j \leq k)$. Similarly, if $x_1, \ldots, x_s$ are loops in $v \in \mathrm{ver}(\mathcal{Q})$, then $\check{x}^k$ stands for a closed path $x_{i_1} \cdots x_{i_k}$ for some $i_1, \ldots, i_k$.

The next two lemmas are well known.

Lemma 2.1. *Suppose $\mathcal{Q}$ is a strongly connected quiver and $\underline{\delta} \in \mathbb{N}^{\#\,\mathrm{arr}(\mathcal{Q})}$. Then the following conditions are equivalent:*

a) There is a closed path h in $\mathcal{Q}$ such that $\mathrm{mdeg}(h) = \underline{\delta}$ and $\mathrm{arr}(h) = \mathrm{arr}(\mathcal{Q})$; in particular, $\mathrm{ver}(h) = \mathrm{ver}(\mathcal{Q})$.

b) We have $\delta_a \geq 1$ for all $a \in \mathrm{arr}(\mathcal{Q})$ and $\sum_{a'=v} \delta_a = \sum_{a''=v} \delta_a$ for all $v \in \mathrm{ver}(\mathcal{Q})$, where the sums range over all $a \in \mathrm{arr}(\mathcal{Q})$ satisfying the given conditions.

We write $\delta(i,j)$ for the Kronecker symbol.

Lemma 2.2. *For positive integers n, d, m the set $\mathcal{Q}(n,d,m)$ is not empty if and only if one of the following possibilities holds:*

a) $n = m = 1$;

b) $n \geq m \geq 2$ and $d \geq n + l - \delta(0,r)$, where $n - 1 = l(m-1) + r$, $l \geq 1$, and $0 \leq r \leq m-2$.

Lemma 2.3. *Suppose $\mathcal{Q}_1, \mathcal{Q}_2$ are strongly connected quivers and $\mathcal{Q}_1 \subset \mathcal{Q}_2$. Then*

$$\#\,\mathrm{arr}(\mathcal{Q}_2) - \#\,\mathrm{arr}(\mathcal{Q}_1) \geq \#\,\mathrm{ver}(\mathcal{Q}_2) - \#\,\mathrm{ver}(\mathcal{Q}_1) + 1.$$

Proof. For every $v \in \mathrm{ver}(\mathcal{Q}_2) \setminus \mathrm{ver}(\mathcal{Q}_1)$ there is an $a \in \mathrm{arr}(\mathcal{Q}_2) \setminus \mathrm{arr}(\mathcal{Q}_1)$ with $a' = v$. There also exists a $b \in \mathrm{arr}(\mathcal{Q}_2) \setminus \mathrm{arr}(\mathcal{Q}_1)$ satisfying $b' \in \mathrm{ver}(\mathcal{Q}_1)$. These remarks imply the required formula. $\square$

2.2 Basic equivalences

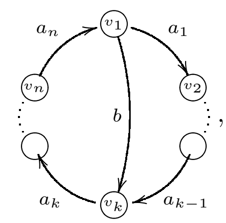
Lemma 2.4. *Suppose* $\mathrm{char}(K) \neq 2$. *If* $\mathcal{Q} \in \mathcal{Q}(n,d,m)$, *$h$ is a closed path in $\mathcal{Q}$, and $h \not\equiv 0$, then* $\deg(h) \leq M(n,d,m)$.

Proof. We claim that $\deg(h) \leq 3n$. If $\deg(h) > 3n$, then there is a vertex $v \in \mathrm{ver}(\mathcal{Q})$ such that $\deg_v(h) \geq 4$. Therefore, $h \equiv h_1 \cdots h_4$ for some closed paths $h_1, \ldots, h_4$ in v. Thus $h \equiv 0$ by the definition of the equivalence $\equiv$; a contradiction.

To complete the proof, it is enough to consider the case of $n = m$ and $d \in \{n, n+1\}$.

1. If $d = n$, then $\mathrm{arr}(\mathcal{Q}) = \{a_1, \ldots, a_n\}$, where $a = a_1 \cdots a_n$ is a primitive closed path. Then $h \equiv a^s$ for some $s > 0$. If $s \geq 3$, then $h \equiv 0$; a contradiction. Thus $\deg(h) \leq 2n$. The case of $n = 1$ and $d = n+1$ can be treated similarly.

2. Let $n = m \geq 2$ and $d = n+1$. In this case $\mathcal{Q}$ is



where $1 \leq k \leq n$. Denote $a = a_1 \ldots a_n$ and

$$c = \left\{ \begin{array}{rl} b, & k = 1 \\ ba_k \ldots a_n, & \text{otherwise} \end{array} \right. .$$

We have $h \equiv a^r c^s$ for some $r, s \geq 0$. If $r = 0$ or $s = 0$, then $\deg(h) \leq 2n$ (see Part 1 of the lemma). Assume that $r, s > 0$. If $r \geq 2$ or $s \geq 2$, then $h \equiv 0$; a contradiction. Hence $\deg(h) = n + \deg c \leq 2n$. □

In what follows we assume that $\mathrm{char}(K) = 2$ unless otherwise stated. We will use the following remark without references to it.

Remark 2.5. *Suppose f, h are closed paths in $\mathcal{Q}$ and b is a subpath of f. Let the equivalence $f \equiv h$ follows from the formulas of the form $a_{\sigma(1)} \cdots a_{\sigma(t)} \equiv a_1 \cdots a_t$, where $a_1, \ldots, a_t$ are closed paths in $v \in \mathrm{ver}(\mathcal{Q})$ satisfying $\deg_v^o(b) = 0$, $t \geq 2$, and $\sigma \in \mathcal{S}_t$. Then b is also a subpath of h.*

There following three lemmas are Lemmas 6.3, 6.8, and 6.9 of [11], respectively.

Lemma 2.6. *Let h be a closed path in $\mathcal{Q}$ and $\{\check{p}\}$ be loops of $\mathcal{Q}$ in some $v \in \mathrm{ver}(\mathcal{Q})$. Then $h \equiv \check{p}^k b$, where $k \geq 0$, $b \in \mathrm{path}(\mathcal{Q})$, and $\deg_{\check{p}}(b) = 0$.*

Moreover, suppose $a \in \mathrm{arr}(h)$ and $a' \neq a''$. If $a' = v$, then $h \equiv a\check{p}^k b_0$; if $a'' = v$, then $h \equiv \check{p}^k a b_0$, where, as above, $\deg_{\check{p}}(b_0) = 0$.

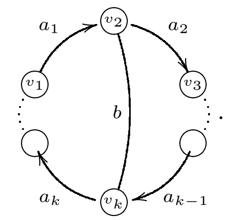
Suppose a quiver $\mathcal{Q}$ contains a path $a = a_1 \cdots a_s$, where $a_1, \ldots, a_s \in \mathrm{arr}(\mathcal{Q})$ are pairwise different. Let h be a closed path in $\mathcal{Q}$ such that $\deg_{a_i}(h) \geq 2$ for all i and there is a $b \in \mathrm{arr}(h)$ satisfying $b \neq a_i$ for all i.

Lemma 2.7. *Using the preceding notation we have $h \equiv a_1 \cdots a_s f$ for some $f \in \mathrm{path}(\mathcal{Q})$. Moreover,*

a) if $b'=a_1''$, then $h\equiv b\,a_1\cdots a_s f$ for some $f\in\mathrm{path}(\mathcal{Q})$;

b) if $b''=a_s'$, then $h\equiv a_1\cdots a_s\,bf$ for some $f\in\mathrm{path}(\mathcal{Q})$.

Let a and h be paths as above. For $1\leq i\leq s$ denote $v_i=a_i''$. We assume that the path a is closed and primitive, $s\geq 2$, $b'\neq b''$, and $b',b''\in\{v_2,v_k\}$ for some $k\in\{1,3,4,\ldots,s\}$. Schematically this is depicted as



Lemma 2.8. *Using the preceding notation we have $h\equiv a_1a_2f_1\,a_1a_2f_2$ for some $f_1,f_2\in\mathrm{path}(\mathcal{Q})$.*

Lemma 2.9. *Suppose $\mathcal{Q}$ is a quiver with n vertices and d arrows. Let h be a closed path in $\mathcal{Q}$ and $h\not\equiv 0$. Then there exist pairwise different primitive closed paths $b_1,\ldots,b_r$, $c_1,\ldots,c_t$ in $\mathcal{Q}$, where $r,t\geq 0$ and $r+t\leq d-n+1$, such that*

$$\mathrm{mdeg}(h)=\sum_{i=1}^{r}\mathrm{mdeg}(b_i)+2\sum_{k=1}^{t}\mathrm{mdeg}(c_k);$$

and there are pairwise different arrows $x_1,\ldots,x_r$, $y_1,\ldots,y_t$, $z_1,\ldots,z_t$ in $\mathcal{Q}$ satisfying

$$y_j,z_j\in\mathrm{arr}(c_j)\ \ \textit{and}\ \ \deg_{y_j}(h)=\deg_{z_j}(h)=2,\tag{2}$$

$$x_i\in\mathrm{arr}(b_i)\ \ \textit{and}\ \ \deg_{x_i}(h)-2\sum_{k=1}^{t}\deg_{x_i}(c_k)=1\tag{3}$$

for any $1\leq i\leq r$, $1\leq j\leq t$.

Proof. The statement of the lemma but the inequality $r+t\leq d-n+1$ follows from Lemma 6.10 [11]. Applying Lemma 2.3, we can assume that $\mathcal{Q}=\mathcal{Q}_{\mathrm{mdeg}\,h}$.

Denote by $\mathcal{G}$ the quiver that is the union of closed paths $b_1,\ldots,b_r$, i.e., $\mathrm{ver}(\mathcal{G})=\mathrm{ver}(b_1)\cup\cdots\cup\mathrm{ver}(b_r)$ and $\mathrm{arr}(\mathcal{G})=\mathrm{arr}(b_1)\cup\cdots\cup\mathrm{arr}(b_r)$. Let $\mathcal{G}_1,\ldots,\mathcal{G}_l$ be the strongly connected components of $\mathcal{G}$. We have $\mathrm{arr}(\mathcal{G}_k)=\bigcup_{i\in I_k}\mathrm{arr}(b_i)$ for some $I_k\subset[1,r]$ and denote $\#I_k=r_k$ $(1\leq k\leq l)$.

We assume that $k=1$. Consider an $i_1\in I_1$ and let $\mathcal{Q}_1$ be the quiver such that $\mathrm{ver}(\mathcal{Q}_1)=\mathrm{ver}(b_{i_1})$ and $\mathrm{arr}(\mathcal{Q}_1)=\mathrm{arr}(b_{i_1})$. If $\#I_1>1$, then there is an $i_2\in I_1\backslash\{i_1\}$ satisfying $\mathrm{ver}(b_{i_2})\cap\mathrm{ver}(\mathcal{Q}_1)\neq\emptyset$. By part a), we have $x\notin\mathrm{arr}(\mathcal{Q}_1)$ for some $x\in\mathrm{arr}(b_{i_2})$. Hence there is an $e_2\in\mathrm{arr}(b_{i_2})$ such that $e_2\notin\mathrm{arr}(\mathcal{Q}_1)$ and $e_2'\in\mathrm{ver}(\mathcal{Q}_1)$. We add the closed path b_{i_2} to $\mathcal{Q}_1$ and obtain a new quiver $\mathcal{Q}_2$, i.e., $\mathrm{ver}(\mathcal{Q}_2)=\mathrm{ver}(\mathcal{Q}_1)\cup\mathrm{ver}(b_{i_2})$ and $\mathrm{arr}(\mathcal{Q}_2)=\mathrm{arr}(\mathcal{Q}_1)\cup\mathrm{arr}(b_{i_2})$. Then we repeat this procedure for $\mathcal{Q}_2$ and so on. Finally, we obtain $\mathcal{Q}_1,\mathcal{Q}_2,\ldots,\mathcal{Q}_{r_1}=\mathcal{G}_1$ and pairwise different arrows $e_2,\ldots,e_{r_1}$ such that $e_j\in\mathrm{arr}(\mathcal{Q}_j)\setminus\mathrm{arr}(\mathcal{Q}_{j-1})$ and $e_j'\in\mathrm{ver}(\mathcal{Q}_{j-1})$ for any $2\leq j\leq r_1$. Then for

the set $V_1 = \{e'_2,\ldots,e'_{r_1}\}$ we have $\#\{a \in \mathrm{arr}(\mathcal{G}_1)\,|\,a' \in V_1\} \geq \#V_1 + (r_1-1)$. Since for every $v \in \mathrm{ver}(\mathcal{G}_1)\backslash V_1$ there is at least one arrow $a \in \mathrm{arr}(\mathcal{G}_1)$ with $a' = v$, we have

$$\#\,\mathrm{arr}(\mathcal{G}_1) \geq \#\,\mathrm{ver}(\mathcal{G}_1) + (r_1-1).$$

The similar formula holds for all k. It follows that

$$\#\,\mathrm{arr}(\mathcal{G}) \geq \#\,\mathrm{ver}(\mathcal{G}) + (r-l). \tag{4}$$

For the quiver $\mathcal{Q}_r = \mathcal{G}$ there is a $j_1 \in [1,t]$ satisfying $\mathrm{ver}(c_{j_1}) \cap \mathrm{ver}(\mathcal{Q}_r) \neq \emptyset$. We add c_{j_1} to $\mathcal{Q}_r$ and denote the resulting quiver by $\mathcal{Q}_{r+1}$. By (2), there exists a $g_1 \in \mathrm{arr}(c_{j_1})$ such that $g_1 \notin \mathrm{arr}(\mathcal{Q}_r)$ and $g'_1 \in \mathrm{ver}(\mathcal{Q}_r)$. Moreover, if the number of strongly connected components of $\mathcal{Q}_{r+1}$ is less than the number of strongly connected components of $\mathcal{Q}_r$, then there also exists a $g_2 \in \mathrm{arr}(c_{j_1})\backslash\{g_1\}$ such that $g_2 \notin \mathrm{arr}(\mathcal{Q}_r)$ and $g'_2 \in \mathrm{ver}(\mathcal{Q}_r)$. We repeat this procedure for $\mathcal{Q}_{r+1}$ and so on. Finally, we obtain quivers $\mathcal{Q}_r, \mathcal{Q}_{r+1},\ldots,\mathcal{Q}_{r+t} = \mathcal{Q}$ and pairwise different arrows $g_1,\ldots,g_{t+l-1}$ of $\mathcal{Q}$ such that for the set $V = \{g'_1,\ldots,g'_{t+l-1}\}$ we have

$$\#\{a \in \mathrm{arr}(\mathcal{Q}) \setminus \mathrm{arr}(\mathcal{G})\,|\,a' \in V\} \geq \#V\backslash\mathrm{ver}(\mathcal{G}) + (t+l-1).$$

Therefore

$$\#\,\mathrm{arr}(\mathcal{Q}) \setminus \mathrm{arr}(\mathcal{G}) \geq \#\,\mathrm{ver}(\mathcal{Q}) \setminus \mathrm{ver}(\mathcal{G}) + (t+l-1)$$

and (4) completes the proof. $\Box$

3 Sets of multidegrees

Suppose $\mathcal{Q}$ is a strongly connected quiver and $\mathrm{char}(K) = 2$.

The *support* of a non-zero vector $\underline{\delta} \in \mathbb{N}^{\#\,\mathrm{arr}(\mathcal{Q})}$ with respect to $\mathcal{Q}$ is the subquiver $\mathcal{Q}_{\underline{\delta}}$ of $\mathcal{Q}$ such that $\mathrm{arr}(\mathcal{Q}_{\underline{\delta}}) = \{a \in \mathrm{arr}(\mathcal{Q}) \mid \delta_a \geq 1\}$ and $\mathrm{ver}(\mathcal{Q}_{\underline{\delta}}) = \{a', a'' \mid a \in \mathrm{arr}(\mathcal{Q}_{\underline{\delta}})\}$. The following remark is extensively applied to established indecomposability of invariants.

Remark 3.1. *Let h be a closed path in $\mathcal{Q}$. If for any* $\mathrm{mdeg}(h)$*-double path a we have that the support of* $\mathrm{mdeg}(h) - 2\,\mathrm{mdeg}(a)$ *is not strongly connected (and is not empty), then $h \not\equiv 0$.*

Proof. If h satisfies the condition of the lemma and $h \equiv 0$, then $h \equiv a^2 f$ for some paths a, f. Thus the support of $\mathrm{mdeg}(h) - 2\,\mathrm{mdeg}(a) = \mathrm{mdeg}(f)$ is strongly connected; a contradiction. $\Box$

For a non-zero vector $\underline{\delta} \in \mathbb{N}^{\#\,\mathrm{arr}(\mathcal{Q})}$ we say that

- $\underline{\delta}$ is *indecomposable* (with respect to $\mathcal{Q}$) if its support is strongly connected;
- $\underline{\delta}$ is *decomposable* (with respect to $\mathcal{Q}$) if its support is not strongly connected but is the disjoint union of strongly connected quivers.

Observe that $\underline{\delta}$ can be neither decomposable nor indecomposable. We say that $\underline{\delta} = \underline{\delta}^{(1)} + \cdots + \underline{\delta}^{(r)}$ is the *decomposition* of $\underline{\delta}$ with respect to $\mathcal{Q}$ if $\underline{\delta}^{(1)},\ldots,\underline{\delta}^{(r)} \in \mathbb{N}^{\#\,\mathrm{arr}(\mathcal{Q})}$ are non-zero vectors and $\mathcal{Q}_{\underline{\delta}^{(1)}},\ldots,\mathcal{Q}_{\underline{\delta}^{(r)}}$ are pairwise different strongly connected components of $\mathcal{Q}_{\underline{\delta}}$. Obviously, if $\underline{\delta}$ is indecomposable, then $r = 1$; and if $\underline{\delta}$ is decomposable, then $r \geq 2$. Introduce the following sets:

a) the set $\Omega_1(\mathcal{Q})$ consists of all $\mathrm{mdeg}(h)$, where h ranges over closed paths in $\mathcal{Q}$ with $\mathrm{arr}(h) = \mathrm{arr}(\mathcal{Q})$;

b) the set $\Omega_2(\mathcal{Q})$ consists of such $\underline{\delta} \in \Omega_1(\mathcal{Q})$ that for every $\underline{\delta}$-double path a in $\mathcal{Q}$ we have $\underline{\delta} - 2\,\mathrm{mdeg}(a)$ is decomposable with respect to $\mathcal{Q}$;

c) the set $\Omega_3(\mathcal{Q})$ consists of such $\underline{\delta} \in \Omega_1(\mathcal{Q})$ that there is no $\underline{\delta}$-double path in $\mathcal{Q}$;

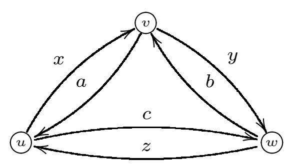
d) the set $\Omega(\mathcal{Q})$ consists of such $\mathrm{mdeg}(h) \in \Omega_1(\mathcal{Q})$ that h is a closed path in $\mathcal{Q}$ and $h \not\equiv 0$.

For every vector $\underline{\delta} \in \Omega_1(\mathcal{Q})$ there exists its decomposition with respect to $\mathcal{Q}$ that consists of one summand. Moreover, by Lemma 2.1, for every $\underline{\theta} \in \Omega_1(\mathcal{Q})$ with $\underline{\delta} - \underline{\theta} \geq 0$ there also exists a decomposition of $\underline{\delta} - \underline{\theta}$ with respect to $\mathcal{Q}$.

Remark 3.2. *We have the following inclusions:* $\Omega_3(\mathcal{Q}) \subset \Omega_2(\mathcal{Q}) \subset \Omega(\mathcal{Q}) \subset \Omega_1(\mathcal{Q})$.

Proof. The inclusion $\Omega_2(\mathcal{Q}) \subset \Omega(\mathcal{Q})$ follows from Remark 3.1. The remaining inclusions are trivial. □

Example 3.3. Let $h_1 = czczxyba$, $h_2 = czcbyzxa$ be closed paths in the quiver $\mathcal{Q}$



Then $h_1 \equiv 0$, $h_2 \not\equiv 0$, and $\mathrm{mdeg}(h_1) = \mathrm{mdeg}(h_2) \in \Omega(\mathcal{Q}) \backslash \Omega_2(\mathcal{Q})$.

Lemma 3.4. *If* $\mathcal{Q} \in \mathcal{Q}(n,d,m)$ *and* $\underline{\delta} \in \Omega_3(\mathcal{Q})$*, then* $|\underline{\delta}| \leq m(d-n+1)$.

Proof. By definition, $\underline{\delta} = \mathrm{mdeg}(h)$ for some closed path h in $\mathcal{Q}$. The definition of $\Omega_3(\mathcal{Q})$ shows that $h \not\equiv 0$. Then Lemma 2.9 implies $\deg(h) \leq m(r+2t)$ and $r + t \leq d - n + 1$. Since $t = 0$, the proof is completed. □

Definition (of a $\underline{\delta}$-complete chain). A *chain of paths* $A = (a_1, \ldots, a_t)$ is an ordered sequence of primitive closed paths satisfying $\mathrm{ver}(a_i) \cap \mathrm{ver}(a_j) = \emptyset$, if $|i-j| > 1$; and $\mathrm{ver}(a_i) \cap \mathrm{ver}(a_j) \neq \emptyset$, otherwise. Given $\underline{\delta} \in \Omega_2(\mathcal{Q})$, the chain of paths A is called $\underline{\delta}$-*complete* if the following holds.

1. The paths $a_1, \ldots, a_t$ are $\underline{\delta}$-double paths.
2. For $\underline{\theta} = \underline{\delta} - 2\sum_{i=1}^{t} \mathrm{mdeg}(a_i)$ we have $\underline{\theta} \geq 0$ and $|\underline{\theta}| > 0$.
3. There is a (unique) decomposition $\underline{\theta} = \underline{\theta}^{(1)} + \cdots + \underline{\theta}^{(r)}$ with respect to $\mathcal{Q}$ and this decomposition satisfies
   a) $r \geq 2$ and $\underline{\theta}^{(i)} \in \Omega_2(\mathcal{Q}_{\underline{\theta}^{(i)}})$ for all i;
   b) if $t \geq 2$, then $r = 2$ and we have $\mathrm{ver}(\mathcal{Q}_{\underline{\theta}^{(i)}}) \cap \mathrm{ver}(a_j) \neq \emptyset$ iff $i = j = 1$ or $i = 2$, $j = t$.

If there is no $\underline{\delta}$-double path in $\mathcal{Q}$, then $A=\emptyset$ is called a $\underline{\delta}$-*complete* chain. Schematically a $\underline{\delta}$-complete chain A is depicted on Figure 1 for $t=1$ and on Figure 2 for $t\geq 2$, where circles stand for closed paths and rectangles stand for subquivers of $\mathcal{Q}$:

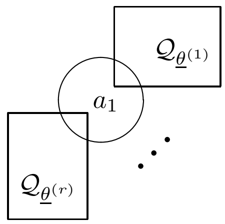


Figure 1.

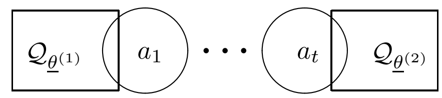
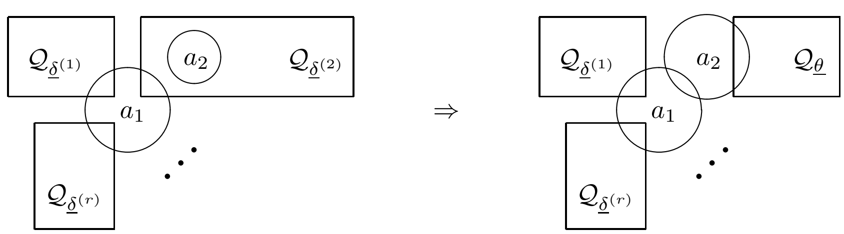


Figure 2.

Lemma 3.5. *For every $\underline{\delta}\in\Omega_2(\mathcal{Q})$ there exists a $\underline{\delta}$-complete chain $A=(a_1,\ldots,a_t)$.*

Proof. If there is no $\underline{\delta}$-double path in $\mathcal{Q}$, then $A=\emptyset$ is a $\underline{\delta}$-complete chain; otherwise, let a_1 be a $\underline{\delta}$-double path in $\mathcal{Q}$. Consider the decomposition $\underline{\delta}-2\,\mathrm{mdeg}(a_1)=\underline{\delta}^{(1)}+\cdots+\underline{\delta}^{(r)}$ with respect to $\mathcal{Q}$. Since $\underline{\delta}\in\Omega_2(\mathcal{Q})$, we have $r\geq 2$. If $\underline{\delta}^{(i)}\in\Omega_2(\mathcal{Q}_{\underline{\delta}^{(i)}})$ for all i, then $A=\{a_1\}$ is a $\underline{\delta}$-complete chain. Thus without loss of generality we can assume that $\underline{\delta}^{(2)}\not\in\Omega_2(\mathcal{Q}_{\underline{\delta}^{(2)}})$, i.e., there exists a $\underline{\delta}^{(2)}$-double path a_2 such that $\underline{\theta}=\underline{\delta}^{(2)}-2\,\mathrm{mdeg}(a_2)$ is indecomposable. But $\underline{\delta}-2\,\mathrm{mdeg}(a_2)$ is decomposable, since $\underline{\delta}\in\Omega_2(\mathcal{Q})$. Hence we obtain $\mathrm{ver}(a_1)\cap\mathrm{ver}(a_2)\neq\emptyset$ and $\mathrm{ver}(a_1)\cap\mathrm{ver}(\mathcal{Q}_{\underline{\theta}})=\emptyset$ (see the picture).



If $r\geq 3$, then we consider a_2 instead of a_1 and obtain that $\underline{\delta}-2\,\mathrm{mdeg}(a_2)=\underline{\theta}'+\underline{\theta}$ is the decomposition of $\underline{\delta}-2\,\mathrm{mdeg}(a_2)$, where $\underline{\theta}'=\underline{\delta}^{(1)}+\underline{\delta}^{(3)}+\cdots+\underline{\delta}^{(r)}+2\,\mathrm{mdeg}(a_1)$ is indecomposable. Thus without loss of generality we can assume that $r=2$.

We have the decomposition $\underline{\delta}-2\,\mathrm{mdeg}(a_1)-2\,\mathrm{mdeg}(a_2)=\underline{\theta}^{(1)}+\underline{\theta}^{(2)}$, where $\underline{\theta}^{(1)}=\underline{\delta}^{(1)}$ and $\underline{\theta}^{(2)}=\underline{\theta}$. Then we consider $\underline{\theta}^{(1)}$ and $\underline{\theta}^{(2)}$ in the same way as we has considered $\underline{\delta}^{(2)}$; and so on. Finally, we obtain a $\underline{\delta}$-complete chain. □

Definition (of a $\underline{\delta}$-tree). For $\underline{\delta}\in\mathbb{N}^{\#\,\mathrm{arr}(\mathcal{Q})}$ a triple $(\mathcal{T},\underline{\delta}^{(v)},A_v\,|\,v\in\mathrm{ver}(\mathcal{T}))$ is called a $\underline{\delta}$-*tree* if the following holds:

1. $\mathcal{T}$ is an oriented rooted tree, i.e., there is no closed path in $\mathcal{T}$, there is a unique $v_0\in\mathrm{ver}(\mathcal{T})$ with $a'\neq v_0$ for all $a\in\mathrm{arr}(\mathcal{T})$, and for each other vertex v of $\mathcal{T}$ there is a unique $a\in\mathrm{arr}(\mathcal{T})$

with $a' = v$. The vertex v_0 is called the *root* and a vertex $v \in \mathrm{ver}(\mathcal{T})$ with $v \neq a''$ for all $a \in \mathrm{arr}(\mathcal{T})$ is called a *leaf*.

2. Suppose $v \in \mathrm{ver}(\mathcal{T})$, then
   a) $\underline{\delta}^{(v)} \in \mathbb{N}^{\#\,\mathrm{arr}(\mathcal{Q})}$ and $\underline{\delta}^{(v_0)} = \underline{\delta}$;
   b) $A_v = (a_1, \ldots, a_t)$ is a $\underline{\delta}^{(v)}$-complete chain;
   c) if $A_v \neq \emptyset$, then $\underline{\delta} - 2\sum_{i=1}^{t} \mathrm{mdeg}(a_i) = \underline{\delta}^{(b'_1)} + \cdots + \underline{\delta}^{(b'_r)}$ is the decomposition with respect to $\mathcal{Q}$, where $b_1, \ldots, b_r$ are all arrows of $\mathcal{T}$ whose tails are equal to v; otherwise v is a leaf.

In particular, the conditions that $v \in \mathrm{ver}(\mathcal{T})$ is a leaf, $A_v = \emptyset$, and $\underline{\delta}^{(v)} \in \Omega_3(\mathcal{Q}_{\underline{\delta}^{(v)}})$ are equivalent. Note that $\#\,\mathrm{ver}(\mathcal{T}) = 1$ iff $\underline{\delta} \in \Omega_3(\mathcal{Q}_{\underline{\delta}})$. By Lemma 3.5, there exists a $\underline{\delta}$-tree for every $\underline{\delta} \in \Omega_2(\mathcal{Q})$. Observe that for different $u, v \in \mathrm{ver}(\mathcal{T})$ and closed paths $a \in A_u$, $b \in A_v$ we have $a \neq b$.

Lemma 3.6. *Suppose $\underline{\delta} \in \Omega_2(\mathcal{Q})\backslash\Omega_3(\mathcal{Q})$ and $(\mathcal{T}, \underline{\delta}^{(v)}, A_v \,|\, v \in \mathrm{ver}(\mathcal{T}))$ is a $\underline{\delta}$-tree. Denote $l = \#\{v \in \mathrm{ver}(\mathcal{T}) \,|\, v \text{ is not a leaf}\}$ and define a set $A = \{a \,|\, a \in A_v \text{ for some } v \in \mathrm{ver}(\mathcal{T})\}$. Then there are pairwise different $c_1, \ldots, c_{l_1} \in A$ such that $A\backslash\{c_1, \ldots, c_{l_1}\} = B_1 \sqcup \cdots \sqcup B_{l_2}$ is a disjoint union, where $B_1, \ldots, B_{l_2}$ are some chains of paths, $0 \le l_1 < l$, and $1 \le l_2 \le l$.*

Proof. We assume that $i = 1$. Suppose $v \in \mathrm{ver}(\mathcal{T})$ is not a leaf, $A_v = (a_1, \ldots, a_t)$, and $b_1, \ldots, b_r$ are arrows of $\mathcal{T}$ whose tails are equal to v. If $t = 1$ and there is a $1 \le j \le r$ such that $A_{b'_j} \neq \emptyset$, then we define $c_i = a_1$, assign b'_j to c_i, and increase i by one.

If $t \ge 2$, then $r = 2$ by the definition of a complete chain. If we also have $A_{b'_1} \neq \emptyset$, then we define $c_i = a_1$, assign b'_1 to c_i, and increase i by one. If $A_{b'_2} \neq \emptyset$, then we define $c_i = a_t$, assign b'_2 to c_i, and increase i by one.

Repeat this procedure for all vertices of $\mathcal{T}$ that are not leaves and obtain a set of pairwise different closed paths $C = \{c_1, \ldots, c_{l_1}\}$. Since we have defined an injection $C \to \{v \in \mathrm{ver}(\mathcal{T}) \,|\, v \text{ is neither a leaf nor the root}\}$, the inequality $l_1 < l$ holds. The claim of the lemma follows from the construction. $\Box$

Lemma 3.7. *Suppose $\mathcal{Q} \in \mathcal{Q}(n, d, m)$ and $A = (a_1, \ldots, a_t)$ is a chain of paths such that for $\underline{\delta} = 2\sum_{i=1}^{t} \mathrm{mdeg}(a_i)$ we have $\underline{\delta} \in \Omega_1(\mathcal{Q})$ and $t \ge 1$. Then $|\underline{\delta}| - mt - n_0 \le n$, where*

$$n_0 = \begin{cases} 0, & \text{if } t = 1 \\ \#\,\mathrm{ver}(a_1) \cap \mathrm{ver}(a_2), & \text{if } t = 2 \\ \#\,\mathrm{ver}(a_2) \cup \cdots \cup \mathrm{ver}(a_{t-1}), & \text{if } t \ge 3 \end{cases}$$

Proof. If $t = 1$, then $\deg(a_1) = n$ and $m = n$. Thus $|\underline{\delta}| - mt - n_0 = n$.

If $t \ge 2$, then $\frac{1}{2}|\underline{\delta}| \le n + n_0$. Therefore $|\underline{\delta}| - mt - n_0 = \sum_{i=1}^{t}(\deg(a_i) - m) + (\frac{1}{2}|\underline{\delta}| - n_0) \le n$, since $\deg(a_i) \le m$. $\Box$

Lemma 3.8. *Suppose $\mathcal{Q} \in \mathcal{Q}(n, d, m)$, $\underline{\delta} \in \Omega_2(\mathcal{Q})$, $A = (a_1, \ldots, a_t) \neq \emptyset$ is a $\underline{\delta}$-complete chain, and $\underline{\theta} = \underline{\delta} - 2\sum_{i=1}^{t} \mathrm{mdeg}(a_i)$. Let $\underline{\theta} = \underline{\theta}^{(1)} + \cdots + \underline{\theta}^{(r)}$ be the decomposition with respect to $\mathcal{Q}$. We define $k = n - \#\,\mathrm{ver}(\mathcal{Q}_{\underline{\theta}})$ and assume that*

$$|\underline{\theta}^{(j)}| \le m(d_j - n_j) + n_j + \rho_j$$

for any $1 \le j \le r$*, where* $d_j = \#\operatorname{arr}(\mathcal{Q}_{\underline{\theta}^{(j)}})$*,* $n_j = \#\operatorname{ver}(\mathcal{Q}_{\underline{\theta}^{(j)}})$*, and* $\rho_j \in \mathbb{Z}$*. Then*

$$|\underline{\delta}| \le m(d-n) + n + \sum_{j=1}^{r} \rho_j + \rho,$$

where $\rho = 2\sum_{i=1}^{t} \deg(a_i) - m(t+1) - k$*.*

Proof. We define a quiver $\mathcal{G}$ by $\operatorname{ver}(\mathcal{G}) = \operatorname{ver}(\mathcal{Q})$ and $\operatorname{ver}(\mathcal{G}) = \operatorname{ver}(\mathcal{Q}_{\theta})$. Let $\mathcal{G}_1, \ldots, \mathcal{G}_l$ be all strongly connected components of $\mathcal{G}$. Then $l = k + r$ and for any $1 \le i \le k+r$ there is an arrow b in $\operatorname{arr}(\mathcal{Q}) \setminus \operatorname{arr}(\mathcal{Q}_{\underline{\theta}})$ such that $b' \in \operatorname{ver}(\mathcal{G}_i)$. Moreover, for any $1 \le i \le t-1$ there are at least two arrows in $\operatorname{arr}(\mathcal{Q}) \setminus \operatorname{arr}(\mathcal{Q}_{\underline{\theta}})$ whose heads are in $\operatorname{ver}(a_i) \cap \operatorname{ver}(a_{i+1})$ and every vertex from $\operatorname{ver}(a_i) \cap \operatorname{ver}(a_{i+1})$ is a strongly connected component of $\mathcal{G}$. These two remarks imply that

$$d \ge \sum_{j=1}^{r} d_j + (k+r) + (t-1).$$

Since $r \ge 2$, we have $\sum_{j=1}^{r} d_j \le d - k - t - 1$ and $\sum_{j=1}^{r} n_j = n - k$. Clearly,

$$|\underline{\delta}| \le m\sum_{j=1}^{r} d_j + (1-m)\sum_{j=1}^{r} n_j + \sum_{j=1}^{r} \rho_j + 2\sum_{i=1}^{t} \deg(a_i),$$

and the above formulas complete the proof. $\square$

Theorem 3.9. *Suppose* $\mathcal{Q} \in \mathcal{Q}(n,d,m)$ *is a quiver and* $\underline{\delta} \in \Omega_2(\mathcal{Q})$*. Then* $|\underline{\delta}| \le m(d-n-1) + 2n$*.*

Proof. If $\underline{\delta} \in \Omega_3(\mathcal{Q})$, then the required formula follows from Lemma 3.4.

Suppose $\underline{\delta} \notin \Omega_3(\mathcal{Q})$ and $(\mathcal{T}, \underline{\delta}^{(v)}, A_v \,|\, v \in \operatorname{ver}(\mathcal{T}))$ is a $\underline{\delta}$-tree. Define the set $I = \{v \in \operatorname{ver}(\mathcal{T}) \,|\, v \text{ is not a leaf}\}$. For $v \in \operatorname{ver}(\mathcal{T})$ denote $m_v = m(\mathcal{Q}_{\underline{\delta}^{(v)}}) \le m$, $n_v = \#\operatorname{ver}(\mathcal{Q}_{\underline{\delta}^{(v)}})$, and $d_v = \#\operatorname{ver}(\mathcal{Q}_{\underline{\delta}^{(v)}})$. If $v \in \operatorname{ver}(\mathcal{T}) \setminus I$, then $\underline{\delta}^{(v)} \in \Omega_3(\mathcal{Q})$ and Lemma 3.4 together with the inequalities $m_v \le m \le n$ and $n_v \le d_v$ implies

$$|\underline{\delta}^{(v)}| \le m_v(d_v - n_v) + m_v \le m(d_v - n_v) + n_v. \tag{5}$$

For $v \in I$ let $A_v = (a_{v1}, \ldots, a_{vt_v})$. We define $\underline{\theta}^{(v)} = \underline{\delta}^{(v)} - 2\sum_{i=1}^{t_v} \operatorname{mdeg}(a_{vi})$ and $k_v = n_v - \#\operatorname{ver}(\mathcal{Q}_{\underline{\theta}^{(v)}})$. By (5), we can apply Lemma 3.8 to all vertices of I starting from elements of the set $\{v \in I \,|\, a' \text{ is a leaf for every } a \in \operatorname{arr}(\mathcal{T}) \text{ with } a'' = v\}$. Hence we obtain

$$|\underline{\delta}| \le m(d-n) + n + \rho,$$

where $\rho = \sum_{v \in I} \rho_v$ and $\rho_v = 2\sum_{i=1}^{t_v} \deg(a_{vi}) - m(t_v + 1) - k_v$.

We consider closed paths $c_1, \ldots, c_{l_1}$ from Lemma 3.6, where $l_1 \le \#I - 1$. For every $v \in I$ we define $J_v \subset [1, t_v]$ by the equality $C_v = A_v \backslash \{c_1, \ldots, c_{l_1}\} = \{a_{vi}\}_{i \in J_v}$ and denote $I_0 = \{v \in I \,|\, C_v \ne \emptyset\}$. Therefore,

$$\rho = \left(2\sum_{v \in I_0} \sum_{i \in J_v} \deg(a_{vi}) - m(t + \#I) - \sum_{v \in I} k_v\right) + 2\sum_{i=1}^{l_1} \deg(c_i),$$

where t stands for $\sum_{v\in I} t_v = l_1 + \sum_{v\in I_0} \#J_v$. Since $\deg(c_i) \leq m$ and $l_1 - \#I \leq -1$, we have

$$\rho \leq \sum_{v\in I_0} \left(2\sum_{i\in J_v} \deg(a_{vi}) - m\#J_v - k_v \right) - m.$$

For all $v \in I_0$ define n_{v0} for the chain of paths C_v in the same way as we have defined n_0 in Lemma 3.7 and let s_v be the number of vertices in C_v. Lemma 3.7 together with the inequality $-k_v \leq -n_{v0}$ implies $\rho \leq \sum_{v\in I_0} s_v - m$. Since there is no $u \in \mathrm{ver}(\mathcal{Q})$ that belongs to C_{v_1} and C_{v_2} for different $v_1, v_2 \in I_0$, we have $\sum_{v\in I_0} s_v \leq n$ and $\rho \leq n - m$. $\Box$

4 Properties of a closed path h with $h \not\equiv 0$

In this section $\mathcal{Q}$ is a strongly connected quiver and $\mathrm{char}(K) = 2$. Let $a = a_1 \cdots a_s$ be a primitive closed path in $\mathcal{Q}$ and $v_1 = a_1'', \ldots, v_s = a_s''$, where $a_1, \ldots, a_s \in \mathrm{arr}(\mathcal{Q})$ and $s \geq 2$. Suppose h is a closed path in $\mathcal{Q}$.

Definition (of good subpaths). A subpath b in h is called *good*, if

a) $b', b'' \in \{v_1, \ldots, v_s\}$;

b) $\deg^o_{v_i}(b) = 0$ for all i;

c) $b \neq a_i$ for all i.

Suppose $h \sim b_1 g_1 b_2 g_2$, where b_1, b_2 are good subpaths in h and g_1, g_2 are paths in $\mathcal{Q}$. Then we say that b_1 and b_2 are *different* subpaths in h.

If we change part c) of the definition of a good path into

c') $b \neq a_i$ for every i satisfying $\deg_{a_i}(h) \leq 2$,

then we obtain the definition of a *semi-good* subpath b in h.

Definition (of good components). A subset $I \subset \{v_1, \ldots, v_s\}$ is called a *good component* with respect to h, if the following conditions are valid:

a) For every good subpath b in h we have $b' \in I$ if and only if $b'' \in I$.

b) There is a good subpath b in h such that $b' \in I$.

c) The set I is a minimal (by inclusion) subset of $\{v_1, \ldots, v_s\}$ that satisfies a) and b).

Taking semi-good subpaths instead of good subpaths in the above definition, we obtain the definition of a *semi-good component*.

Let $I_1, \ldots, I_r$ be all good components with respect to h. Obviously,

$$\{v_1, \ldots, v_s\} = I_0 \sqcup I_1 \sqcup \cdots \sqcup I_r \qquad (6)$$

for some $I_0 \subset \{v_1, \ldots, v_s\}$. Formula (6) is called the *decomposition into good components* with respect to h and I_0 is called the *null component* with respect to h.

In what follows we assume that $\deg_{a_i}(h) = 2$ for all i unless otherwise stated.

Lemma 4.1. *1. For every good subpath b in h we have $b' \notin I_0$ and $b'' \notin I_0$.*

2. For all $u, w \in I_j$, where $j > 0$, there are pairwise different good subpaths $b_1, \ldots, b_l$ in h such that $b_1 \cdots b_l$ is a closed path in $\mathcal{Q}$, $b_1'' = u$, and $b_k' = w$ for some k with $1 \leq k \leq l$.

Proof. Part 1 follows from the definition. Let $\mathcal{G}$ be the h-restriction of $\mathcal{Q}$ to the vertices $v_1, \ldots, v_s$ (see Section 2.1 for the definition). We consider h as a path in $\mathcal{G}$ and define $\underline{\theta} \in \mathbb{N}^{\mathrm{arr}(\mathcal{G})}$ as $\underline{\theta} = \mathrm{mdeg}(h) - 2\,\mathrm{mdeg}(a)$. Since $\deg_{a_i}(h) = 2$ for all i, it is not difficult to see that for the decomposition $\underline{\theta} = \underline{\theta}^{(1)} + \cdots + \underline{\theta}^{(r)}$ with respect to $\mathcal{Q}_{\underline{\theta}}$ we have $\mathrm{ver}(\mathcal{Q}_{\underline{\theta}^{(j)}}) = I_j$ for any $1 \leq j \leq r$. To conclude the proof, we apply Lemma 2.1 to $\underline{\theta}^{(j)}$ and $\mathcal{Q}_{\underline{\theta}^{(j)}}$. □

Lemma 4.2. *If $h \not\equiv a^2 f$ for all $f \in \mathrm{path}(\mathcal{Q})$, then the number of good components with respect to h is equal or greater than two.*

Proof. Let r be the number of good components and $\underline{\delta} = \mathrm{mdeg}(h)$. If $r = 0$, then

$$\delta_b = \begin{cases} 0, & \text{if } b \neq a_i \text{ for all } i \\ 2, & \text{otherwise} \end{cases}$$

for $b \in \mathrm{arr}(\mathcal{Q})$. Hence $h \sim a^2$ and we have a contradiction.

Suppose $r = 1$. If $v_i \in I_0$, then $h \sim a_{i-1} a_i f_1 a_{i-1} a_i f_2$ for some paths f_1, f_2 that do not contain a_{i-1} and a_i. Substitute a new arrow a_{s+1} for the path $a_{i-1} a_i$. Repeat this procedure for all elements of I_0. Thus we can assume that $I_0 = \emptyset$ and $I = \{v_1, \ldots, v_s\}$ is the only good component.

If $s = 1$, then Lemma 2.6 implies a contradiction. Otherwise, we consider the h-restriction of $\mathcal{Q}$ to $v_1, \ldots, v_s$, remove arrows $a_1, \ldots, a_s$ from this restriction, and denote the resulting quiver by $\mathcal{G}$. Let $\mathcal{T}$ be a spanning tree for $\mathcal{G}$, i.e.,

a) $\mathrm{ver}(\mathcal{T}) = \{v_1, \ldots, v_s\}$ and $\mathrm{arr}(\mathcal{T}) \subset \mathrm{arr}(\mathcal{G})$;

b) If we consider $\mathcal{T}$ as a graph without orientation, then it is a tree.

Consider a leaf v_i of $\mathcal{T}$ together with the unique arrow $b \in \mathrm{arr}(\mathcal{T})$ satisfying $v_i \in \{b', b''\}$. Then the condition of Lemma 2.8 is true and we have $h \equiv a_{i-1} a_i f_1\, a_{i-1} a_i f_2$ for some $f_1, f_2 \in \mathrm{path}(\mathcal{Q})$. We remove the vertex v_i and the arrow b from $\mathcal{T}$ and denote the resulting quiver by $\mathcal{T}_1$. As above, we consider some leaf of $\mathcal{T}_1$, apply Lemma 2.8, and so on. Finally, we obtain $h \equiv a f_1 a f_2 \equiv a^2 f_1 f_2$ for some paths $f_1, f_2 \in \mathrm{path}(\mathcal{Q})$; a contradiction. □

Lemma 4.3. *Suppose $\{v_1, \ldots, v_s\} = I_0 \sqcup I_1 \sqcup \cdots \sqcup I_r$ is the decomposition into good components with respect to h, $r \geq 2$, and $V \subset \{v_1, \ldots, v_s\} \backslash I_1$. Let b, c, e be pairwise different good subpaths in h such that*

a) $b' \in I_1$ and $c', e' \in V$;

b) $v \in \mathrm{ver}(b) \cap \mathrm{ver}(c) \cap \mathrm{ver}(e)$ for some v.

Then there exists a closed path h_0 in $\mathcal{Q}$ such that $h_0 \equiv h$ and

$$\{v_1,\ldots,v_s\} = I_0 \sqcup \overline{I}_1 \sqcup \bigsqcup_{k\in D} I_k \sqcup J_1 \sqcup \cdots \sqcup J_l$$

is the decomposition into good components with respect to h_0, where $l \geq 0$ and $D = \{2,\ldots,r\}\backslash\{i,j\}$ for $c' \in I_i$, $e' \in I_j$. Moreover, $\#\overline{I}_1 > \#I_1$.

Proof. We have $b = b_1b_2$, $c = c_1c_2$, and $e = e_1e_2$ for some paths b_i, c_i, e_i in $\mathcal{Q}$ ($i = 1, 2$) with $b_1' = c_1' = e_1' = v$. There are two possibilities:

1. If $h \sim b_2f_1c_1 \cdot c_2f_2e_1 \cdot e_2f_3b_1$ for some $f_1, f_2, f_3 \in \mathrm{path}(\mathcal{Q})$, then we define $h_0 = b_2f_1c_1 \cdot e_2f_3b_1 \cdot c_2f_2e_1$ and we have $h_0 \equiv h$. Let S_0 and S be the sets of good subpaths in h_0 and h, respectively. Then $S_0 = (S \cup \{b_1c_2, c_1e_2, e_1b_2\})\backslash\{b, c, e\}$. Clearly, I_k is a good component with respect to h_0, where $2 \leq k \leq r$ and $k \neq i, j$, and I_0 is the null component with respect to h_0. By part 2 of Lemma 4.1, the set I_1 and the vertices c' and e'' belong to one and the same good component with respect to h_0, which we denote by $\overline{I}_1$. Thus $\#\overline{I}_1 > \#I_1$ and the claim is proven.

2. If $h \sim b_2f_1e_1 \cdot e_2f_2c_1 \cdot c_2f_3b_1$ for some $f_1, f_2, f_3 \in \mathrm{path}(\mathcal{Q})$, then the proof is analogous. □

Lemma 4.4. *Suppose $h \not\equiv a^2f$ for all $f \in \mathrm{path}(\mathcal{Q})$. Then there exists a closed path h_0 in $\mathcal{Q}$ and a good component I with respect to h_0 such that $h_0 \equiv h$ and if good subpaths b, c in h_0 and $v \in \mathrm{ver}(\mathcal{Q})$ satisfy the following condition:*

$$b' \in I,\ c' \notin I,\ \text{ and } v \in \mathrm{ver}(b) \cap \mathrm{ver}(c), \tag{7}$$

then

- *a) b is the unique good subpath in h_0 satisfying (7), i.e., $h_0 \sim bf_0$, where f_0 do not contain a good subpath b_1 with $b_1' \in I$ and $v \in \mathrm{ver}(b_1)$;*
- *b) $\deg_v(b) = 1$.*

Proof. The proof consists of two parts. At first we find h_0 and I that satisfy condition a), then we change h_0 to make condition b) valid.

a) For a good component I with respect to h and $V \subset \{v_1,\ldots,v_s\}\backslash I$, we write $I > V$ if the condition of Lemma 4.3 does not hold for $I_1 = I$ and V.

Suppose $\{v_1,\ldots,v_s\} = I_0 \sqcup I_1 \sqcup \cdots \sqcup I_r$ is the decomposition into good components with respect to h. If $I_1 \not> I_2 \sqcup \cdots \sqcup I_r$, then Lemma 4.3 implies that there is an $h^{(0)} \equiv h$ such that $\#\overline{I}_1 > \#I_1$ for a good component $\overline{I}_1$ with respect to $h^{(0)}$. Repeat this procedure for $\overline{I}_1$ and so on. Finally, we obtain $h_1 \equiv h$ such that $I_{11},\ldots,I_{1r_1}$ are all good components with respect to h_1 and $I_{11} > I_{12} \sqcup \cdots \sqcup I_{1r_1}$. Note that $r_1 \geq 2$ by Lemma 4.2.

If $I_{12} \not> I_{13} \sqcup \cdots \sqcup I_{1r_1}$, then we act as above; and so on. Finally, we obtain $h_l \equiv h$ such that $I_{l1},\ldots,I_{lr_l}$ are all good components with respect to h_l and $I_{ii} > I_{i,i+1} \sqcup \cdots \sqcup I_{ir_l}$ for any $1 \leq i < r_l$. Then condition a) holds for $h_0 = h_l$ and $I = I_{lr_l}$.

b) Consider h_0 and I that have been constructed in part a) of the proof. Suppose b, c are good subpaths with respect to h_0, $b' \in I$, $c' \notin I$, and $v \in \mathrm{ver}(\mathcal{Q})$. If $\deg_v(b) \geq 2$, then $b = b_1qb_2$ for some paths b_1, q, b_2 satisfying $q' = q'' = v$ and $\deg_v(b_1b_2) = 0$. Assume that $c = c_1c_2$ for paths c_1, c_2 with $c_1' = c_2'' = v$ and $h \sim bf_1cf_2$ for some paths f_1, f_2. Then

$$h_0 \sim b_2f_1c_1 \cdot c_2f_2b_1 \cdot q \equiv c_2f_2b_1 \cdot b_2f_1c_1 \cdot q,$$

and we define $h_1 = c_2 f_2 b_1 \cdot b_2 f_1 c_1 \cdot q$. Let S_1 and S_0 be the sets of good subpaths in h_1 and h_0, respectively. Then $S_1 = (S_0 \cup \{b_1 b_2, c_1 q c_2\}) \backslash \{b, c\}$. It is not difficult to see that every good component with respect to h_1 is a good component with respect to h_0 and vice versa. Moreover, condition a) remains valid for h_1.

If condition b) of the lemma does not hold for h_1 and some paths b and c, then we repeat the above procedure for h_1; and so on. Denote by k the sum $\sum \deg b$ that ranges over all $b \in \mathrm{arr}(\mathcal{Q})$ with $b' \in I$. After each step of the procedure k is diminished by a positive number. Hence we finally obtain h_0 that satisfies conditions a) and b). $\square$

Now we assume that h is a closed path in $\mathcal{Q}$ with $\deg_{a_i}(h) \geq 2$ for all i.

Lemma 4.5. *Suppose $h \not\equiv a^2 f$ for all $f \in \mathrm{path}(\mathcal{Q})$. Then there exists a closed path h_0 in $\mathcal{Q}$ and a semi-good component I with respect to h_0 such that $h_0 \equiv h$ and if good subpaths b, c in h_0 and $v \in \mathrm{ver}(\mathcal{Q})$ satisfy (7) then conditions a) and b) of Lemma 4.4 are valid.*

Proof. Suppose $h \sim a_i c_1 \cdots a_i c_l$ for some $1 \leq i \leq s$, where $l = \deg_{a_i}(h) \geq 3$. Then we add a new arrow b_i to $\mathcal{Q}$ and define $b_i' = a_i'$, $b_i'' = a_i''$. Moreover, we substitute $a_i c_1 a_i c_2 b_i c_3 \cdots b_i c_l$ for h. After performing this procedure for all i we obtain a strongly connected quiver $\mathcal{G}$ and a closed path h_1 in $\mathcal{G}$ satisfying $\deg_{a_i}(h_i) = 2$ for all i. Lemma 4.4 completes the proof. $\square$

5 The main upper bound

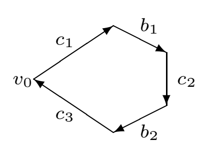
Suppose $\mathcal{Q} \in \mathcal{Q}(n, d, m)$ is a quiver and $\mathrm{char}(K) = 2$. The set $\Omega_2(\mathcal{Q})$ has been defined in Section 3. This section is dedicated to the proof of the following theorem.

Theorem 5.1. *If $h \not\equiv 0$ is a closed path in $\mathcal{Q}$, then $\deg(h) \leq m(d - n - 1) + 2n$.*

Lemma 5.2. *Suppose h is a closed path in $\mathcal{Q}$ such that $h \not\equiv 0$ and $\mathrm{mdeg}(h) \notin \Omega_2(\mathcal{Q})$. Then $h \equiv cf$, where f and $c = c_1 b_1 c_2 b_2 c_3$ are closed paths in $\mathcal{Q}$, paths c_1, c_2, c_3 can be empty, $b_1, b_2 \in \mathrm{arr}(\mathcal{Q})$, and the following conditions hold:*

a) $\deg_{b_1}(h) = \deg_{b_2}(h) = 1$*;*

b) $\mathrm{ver}(c_1) \cap \mathrm{ver}(c_2) = \emptyset$*,* $\mathrm{ver}(c_2) \cap \mathrm{ver}(c_3) = \emptyset$*, and* $\mathrm{ver}(c_1) \cap \mathrm{ver}(c_3) = c_1'' = c_3' = v_0$*. Schematically, this condition is depicted as*



c) for all $v \in \mathrm{ver}(c)$ with $\deg_v(c) \geq 2$ we have $\deg_v(c) = \deg_v(h)$. In particular, $\deg_{v_0}(c) = 1$;

d) for all $v \in \mathrm{ver}(c_2)$ we have $\deg_v(h) > \deg_v(c) = 1$.

Proof. Since $\mathrm{mdeg}(h) \notin \Omega_2(\mathcal{Q})$, there is a $\mathrm{mdeg}(h)$-double path a in $\mathcal{Q}$ such that $\mathrm{mdeg}(\mathcal{Q}) - 2\,\mathrm{mdeg}(a)$ is indecomposable. We apply Lemma 4.5 to h and a to obtain a closed path h_0 in $\mathcal{Q}$ and a semi-good component I satisfying the conditions from Lemma 4.5. Without loss of generality we can assume that $h = h_0$. In what follows, all good subpaths in h will be considered with respect to a. Define the subset

$$V \subset \mathrm{ver}(\mathcal{Q}) \setminus \mathrm{ver}(a)$$

that contains v if and only if there is a good subpath e in h such that $e' \notin I$ and $v \in \mathrm{ver}(e)$. Since $\mathrm{mdeg}(\mathcal{Q}) - 2\,\mathrm{mdeg}(a)$ is indecomposable, there is a good subpath $b = b_1 \cdots b_l$ in h satisfying $b' \in I$ and $\mathrm{ver}(b) \cap V \neq \emptyset$, where $b_1, \ldots, b_l \in \mathrm{arr}(\mathcal{Q})$. Since $b_1'' \notin V$ and $b_l' \notin V$, we can define

$$i = \min\{1 \leq k \leq l \,|\, b_k' \in V\} \text{ and } j = \min\{i < k \leq l \,|\, b_k' \notin V\}.$$

Let $\deg_{b_i}(h) > 1$, then there is a good subpath e in h with $b_i \in \mathrm{arr}(e)$ and $h \sim b f_1 e f_2$ for some $f_1, f_2 \in \mathrm{path}(\mathcal{Q})$ because $\deg_{b_i'}(b) = 1$ by Lemma 4.5. If $e' \in I$, then we obtain a contradiction to Lemma 4.5. If $e' \notin I$, then $b_i'' \in V$; a contradiction. Therefore, $\deg_{b_i}(h) = 1$ and, similarly, $\deg_{b_j}(h) = 1$.

If $\mathrm{ver}(b_1 \cdots b_{i-1}) \cap \mathrm{ver}(b_{j+1} \cdots b_l) \neq \emptyset$, then $h \sim cf$ for $c = c_1 b_i c_2 b_j c_3$ satisfying part b) of the lemma, where f is a path in $\mathcal{Q}$, $c_2 = b_{i+1} \cdots b_{j-1} \in \mathrm{path}(\mathcal{Q})$, and $c_1, c_3 \in \mathrm{path}(\mathcal{Q})$ are subpaths in $b_1 \cdots b_{i-1}$, $b_{j+1} \cdots b_l$, respectively. Moreover, we can assume that $\deg_{v_0}(c_1) = \deg_{v_0}(c_3) = 1$.

If $\mathrm{ver}(b_1 \cdots b_{i-1}) \cap \mathrm{ver}(b_{j+1} \cdots b_l) = \emptyset$, then, taking into account Lemma 2.7, we have $h \equiv cf$, where c satisfies the same conditions as above and f is a path in $\mathcal{Q}$.

If there is a $v \in \mathrm{ver}(c_1)$ such that $\deg_v(h) > \deg_v(c_1) \geq 2$, then $c_1 = e_1 e_2 e_3$, where $e_1, e_2, e_3 \in \mathrm{path}(\mathcal{Q})$, $e_2' = e_2'' = v$, and $\deg_v(e_1) = \deg_v(e_3) = 1$. We have $h \sim e_1 \cdot e_2 \cdot e_3 b_i c_2 b_j c_3 f_1 \cdot p \cdot f_2$ for some $f_1, f_2 \in \mathrm{path}(\mathcal{Q})$ and $p' = p'' = v$. Thus $h \equiv e_1 \cdot e_3 b_i c_2 b_j c_3 f_1 \cdot e_2 \cdot p \cdot f_2$ and we change notations by putting $c = e_1 e_3 b_i c_2 b_j c_3$. We repeat this procedure for all vertices of c_1, c_2, c_3 and obtain that part c) of the lemma holds.

Since $\mathrm{ver}(c_2) \subset V$, part d) of the lemma is a consequence of part c). $\square$

We assume that $h \equiv cf \not\equiv 0$ is a closed path from Lemma 5.2, where $c = c_1 b_1 c_2 b_2 c_3$. Define sets

$$S_{arr} = \{a \in \mathrm{arr}(c_3 c_1) \mid \deg_a(h) = \deg_a(c)\},$$
$$S_{ver} = \{v \in \mathrm{ver}(c_3 c_1) \mid \deg_v(h) = \deg_v(c)\}.$$

In this section we will use the next remark.

Remark 5.3.

1. *Since f is not an empty path, we have $v_0 \notin S_{ver}$.*
2. *For all $v \in \mathrm{ver}(c_i)$, $a \in \mathrm{arr}(c_i)$ the equalities $\deg_v(c) = \deg_v(c_i)$, $\deg_a(c) = \deg_a(c_i)$ hold ($i = 1, 2, 3$).*

Lemma 5.4. *For all $v \in \mathrm{ver}(c_3 c_1)$, $a \in \mathrm{arr}(c_3 c_1)$ we have*

a) if $a' \in S_{ver}$ or $a'' \in S_{ver}$, then $a \in S_{arr}$;

b) if $a' \notin S_{ver}$ or $a'' \notin S_{ver}$, then $\deg_a(c) = 1$;

c) if $v \notin S_{ver}$, then $\deg_v(c) = 1$.

Proof. Part a) is trivial. Part c) follows from part c) of Lemma 5.2. If $a' \notin S_{ver}$, then $\deg_{a'}(c) = 1$ by part c); hence $\deg_a(c) = 1$ and part b) is proven. □

The following lemma will help us to perform the induction step in the proof of Theorem 5.1.

Lemma 5.5. *We assume that for every strongly connected quiver $\mathcal{G}$ with $\#\operatorname{arr}(\mathcal{G}) < d$ the assertion of Theorem 5.1 is valid. Then* $\deg(c) \leq m(\#S_{arr} - \#S_{ver} + 1) + 2\#S_{ver}$.

Proof. Let $\#S_{ver} = \emptyset$. Then c is a primitive closed path by parts c) and d) of Lemma 5.2. Hence $\deg(c) \leq m$.

Let $\#S_{ver} \geq 1$. Using the fact that $v_0 \notin S_{ver}$ and part b) of Lemma 5.2 we obtain

$$S_{ver} = S^1_{ver} \sqcup S^3_{ver} \text{ and } S_{arr} = S^1_{arr} \sqcup S^3_{arr} \tag{8}$$

for $S^1_{arr} = S_{arr} \cap \operatorname{arr}(c_1)$, $S^3_{arr} = S_{arr} \cap \operatorname{arr}(c_3)$, $S^1_{ver} = S_{ver} \cap \operatorname{ver}(c_1)$, and $S^3_{ver} = S_{ver} \cap \operatorname{ver}(c_3)$.

Suppose $\#S^1_{ver} \geq 1$. We assume that $c_1 = x_1 \cdots x_s$ for $x_1, \ldots, x_s \in \operatorname{arr}(\mathcal{Q})$, where for $i \neq j$ we can have $x_i = x_j$. We define $p = \min\{1 \leq k \leq s \,|\, x'_k \in S^1_{ver}\}$ and $q = \max\{1 \leq k \leq s \,|\, x'_k \in S^1_{ver}\}$. Then $c_1 = e_1e_2e_3$, where paths $e_1 = x_1 \cdots x_p$, $e_2 = x_{p+1} \cdots x_q$, and $e_3 = x_{q+1} \cdots x_s$ can be empty. We claim that

$$\deg(e_2) \leq m(\#S^1_{arr} - \#S^1_{ver}) + 2\#S^1_{ver}. \tag{9}$$

To prove the claim we consider the e_2-restriction of $\mathcal{Q}$ to S^1_{ver}, add a new arrow z from e'_2 to e''_2, and denote the resulting quiver by $\mathcal{G}$. In other words, $\operatorname{ver}(\mathcal{G}) = S^1_{ver}$ and $a \in \operatorname{arr}(\mathcal{G})$ has one of the following types:

1. $a = \widetilde{x_i}$, where $1 \leq i \leq s$ and $x'_i, x''_i \in S^1_{ver}$;
2. $a = \widetilde{x_i \cdots x_j}$ for $1 \leq i < j \leq s$, where $x''_i, x'_j \in S_{ver}$ and $x'_i, \ldots, x'_{j-1} \notin S_{ver}$;
3. $a = z$.

Note that for an arrow $a = \widetilde{x_i}$ of type 1 we have $x_i \in S_{arr}$ by part a) of Lemma 5.4 and we say that x_i is assigned to a. Similarly, for an arrow $a = \widetilde{x_i \cdots x_j}$ of type 2 we have $x_i, x_j \in S_{arr}$ and we say that x_i, x_j are assigned to a; moreover,

a) $\deg_{x_k}(e_2) = 1$ for any $i \leq k \leq j$ (see part b) of Lemma 5.4).

b) $\deg_{x'_k}(c) = \deg_{x'_k}(e_2) = 1$ for any $i \leq k \leq j-1$ (see part c) of Lemma 5.4). In particular, $x_i \cdots x_j$ is either a primitive closed path or it is a subpath of c without self-intersections; thus, $\deg(x_i \cdots x_j) \leq m$.

Let y be the unique path in $\mathcal{G}$ that corresponds to the path e_2 in $\mathcal{Q}$. The quiver $\mathcal{G}$ is strongly connected, since yz is a closed path in $\mathcal{G}$ that contains all arrows and all vertices of $\mathcal{G}$. Moreover, we have $yz \not\equiv 0$, since $\deg_a(y) = 1$ for every arrow a of type 2, $\deg_z(y) = 0$, and $h \not\equiv 0$.

For every arrow a of type 1 there is an arrow from S^1_{arr} that is assigned to a; and for every arrow b of type 2 there are two arrows from S^1_{arr} that are assigned to b. But the arrow $x_p \in S^1_{arr}$ is not assigned to any arrow of $\mathcal{G}$. Therefore,

$$\#\operatorname{arr}(\mathcal{G}) - 1 \leq \#S^1_{arr} - l - 1,$$

where l stands for the number of arrows of type 2. Since $b_1, b_2 \notin S^1_{arr}$, it follows that $\#\,\mathrm{arr}(\mathcal{G}) \leq \#S^1_{arr} < \mathrm{arr}(\mathcal{Q}) = d$. Applying Theorem 5.1 to $\mathcal{G}$, we obtain

$$\deg(yz) \leq m(\mathcal{G})(\#\,\mathrm{arr}(\mathcal{G}) - \#\,\mathrm{ver}(\mathcal{G})) + 2\#\,\mathrm{ver}(\mathcal{G})\,.$$

It is not difficult to see that $m(\mathcal{G}) \leq m$. Thus

$$\deg(y) \leq m(\#S^1_{arr} - \#S^1_{ver} - l) + 2\#S^1_{ver} - 1.$$

By property b) of paths of type 2, we have

$$\deg(e_2) \leq \deg(y) + l(m-1).$$

The last two formulas conclude the proof of (9).

If $\#S^3_{ver} \geq 1$, then we rewrite c_3 in a form $c_3 = g_1 g_2 g_3$ in the same way as we have done for $c_1 = e_1 e_2 e_3$. Then the proof of the formula

$$\deg(g_2) \leq m(\#S^3_{arr} - \#S^3_{ver}) + 2\#S^3_{ver} \tag{10}$$

is similar to the proof of (9).

Suppose $S^1_{ver} \neq \emptyset$ and $S^3_{ver} \neq \emptyset$. Then

$$\deg(c) = \deg(c_1 b_1 c_2 b_2 c_3) = \deg(e_2) + \deg(g_2) + \deg(f_1) + \deg(f_2),$$

where $f_1 = g_3 e_1$ and $f_2 = e_3 b_1 c_2 b_2 g_1$. Parts c) and d) of Lemma 5.2 imply that

a) for every $v \in (\mathrm{ver}(f_1) \cup \mathrm{ver}(f_2)) \backslash \{f_1', f_1'', f_2', f_2''\}$ we have $\deg_v(c) = 1$;

b) $(\mathrm{ver}(f_1) \cup \mathrm{ver}(f_2)) \cap (\mathrm{ver}(e_2) \cup \mathrm{ver}(g_2)) = \{f_1', f_1'', f_2', f_2''\}$.

It follows that there are paths d_1, d_2 in $\mathcal{Q}$ such that $f_1 d_1 f_2 d_2$ is a primitive closed path in $\mathcal{Q}$. In particular, $\deg(f_1) + \deg(f_2) \leq m$. Formulas (9) and (10) conclude the proof of the lemma.

The cases $S^1_{ver} \neq \emptyset, S^3_{ver} = \emptyset$ and $S^1_{ver} = \emptyset, S^3_{ver} \neq \emptyset$ can be treated in the similar fashion. If $S^1_{ver} = \emptyset$ and $S^3_{ver} = \emptyset$, then $S_{ver} = \emptyset$; a contradiction. □

Proof of Theorem 5.1. Suppose $\mathcal{Q}_{\mathrm{mdeg}(h)} \in \mathcal{Q}(n_0, d_0, m_0)$ for some n_0, d_0, m_0. We assume that the theorem is proven for the case $\mathcal{Q} = \mathcal{Q}_{\mathrm{mdeg}(h)}$. Then we have $\deg(h) \leq m_0(d_0 - n_0 - 1) + 2n_0$. Lemma 2.3 implies

$$\deg(h) \leq m(d_0 - n_0) + 2n_0 - m_0 \leq m(d - n - 1) + 2n - m_0$$

and we obtain the required upper bound. Therefore, without loss of generality we can assume that $\mathcal{Q} = \mathcal{Q}_{\mathrm{mdeg}(h)}$.

We prove the theorem by induction on $\#\,\mathrm{arr}(\mathcal{Q})$.

Induction base. If $\#\,\mathrm{arr}(\mathcal{Q}) = 1$, then $\mathrm{ver}(\mathcal{Q}) = \{v\}$ and the only arrow of $\mathcal{Q}$ is a loop in v. Then $\deg(h) = 1$ and the required upper bound on $\deg(h)$ holds.

Induction step. If $\mathrm{mdeg} \in \Omega_2(\mathcal{Q})$, then see Theorem 3.9; otherwise we apply Lemma 5.2 to h and obtain $h \equiv cf$, where f and $c = c_1 b_1 c_2 b_2 c_3$ are closed paths in some vertex v_0. By Lemma 5.5, we have

$$\deg(c) \leq m(\#S_{arr} - \#S_{ver} + 1) + 2\#S_{ver}. \tag{11}$$

We define the quiver $\mathcal{G}$ by $\mathrm{ver}(\mathcal{G}) = \{v \in \mathrm{ver}(\mathcal{Q}) \mid \deg_v(h) > \deg_v(c)\}$ and $\mathrm{arr}(\mathcal{G}) = \{a \in \mathrm{arr}(\mathcal{Q}) \mid \deg_a(h) > \deg_a(c)\} \bigcup \{x\}$, where x is a new loop in the vertex v_0. Then xf is a closed path in $\mathcal{G}$ that contains all vertices and arrows of $\mathcal{G}$. In particular, $\mathcal{G}$ is a strongly connected quiver and $\mathcal{G} = \mathcal{G}_{\mathrm{mdeg}(xf)}$. Since $cf \not\equiv 0$, we have $xf \not\equiv 0$. By parts a) and d) of Lemma 5.2, we have $\#\,\mathrm{arr}(\mathcal{G}) \leq d - \#S_{arr} - 1$ and $\mathrm{ver}(\mathcal{G}) = n - \#S_{ver}$. Applying induction hypothesis to the closed path xf in $\mathcal{G}$ and using the inequalities $\mathrm{arr}(\mathcal{G}) > \mathrm{ver}(\mathcal{G})$ and $m(\mathcal{G}) \leq m$, we obtain

$$\deg(xf) \leq m(d - n - 1) + 2n - m(\#S_{arr} - \#S_{ver} + 1) - 2\#S_{ver}.$$

Formula (11) implies the required upper bound on $\deg(h)$. $\square$

6 The upper bound for the case of small d

Assume that $\mathrm{char}(K) = 2$. The following lemma is a stronger version of Lemma 2.9.

Lemma 6.1. *Suppose $\mathcal{Q} \in \mathcal{Q}(n, d, m)$. Then using the notation of Lemma 2.9 we have $r \geq 1$.*

Proof. Suppose $r = 0$. Then $\mathrm{mdeg}(h) = 2\sum_{j=1}^{t} \mathrm{mdeg}(c_j)$, where $t \geq 1$, and we have two possibilities.

1. Let $\mathrm{mdeg}(h) \notin \Omega_2(\mathcal{Q})$. Then there exists a primitive closed path $a = a_1 \cdots a_s$ in $\mathcal{Q}$ ($a_1, \ldots, a_s \in \mathrm{arr}(\mathcal{Q})$) such that $\mathrm{mdeg}(h) - 2\,\mathrm{mdeg}(a)$ is indecomposable and $\deg_{a_i}(h) \geq 2$ for all i. It is not difficult to see that $\mathrm{ver}(a) = I \sqcup J$, where

1) $\deg_v(h) = 2$ for all $v \in I$;

2) for every $u, v \in J$ with $u \neq v$ there is a path g in $\mathcal{Q}$ from u to v; moreover, for every $e \in \mathrm{arr}(g)$ we have $\deg_e(h) \geq 2$, if $e \notin \mathrm{arr}(a)$; and $\deg_e(h) \geq 4$, if $e \in \mathrm{arr}(a)$. Lemma 2.7 implies that $h \equiv gf$ for some path f.

If $s > 1$, then, applying Lemma 2.8, we have $h \equiv a_1 a_2 f_1 a_1 a_2 f_2 \equiv a_1 a_2 a_3 f_3 a_1 a_2 a_3 f_4 \equiv \cdots \equiv a f_{2s-3} a f_{2s-2}$ for some paths $f_1, \ldots, f_{2s-2}$. Lemma 2.6 gives $h \equiv 0$ for $s \geq 1$; a contradiction.

2. If $\mathrm{mdeg}(h) \in \Omega_2(\mathcal{Q})$, then we consider a $\mathrm{mdeg}(h)$-tree $(\mathcal{T}, \underline{\delta}^{(v)}, A_v \mid v \in \mathrm{ver}(\mathcal{T}))$ constructed in Section 3. For a leaf $v \in \mathrm{ver}(\mathcal{T})$ we have $\underline{\delta}^{(v)} \in \Omega_3(\mathcal{Q}_{\underline{\delta}^{(v)}})$; a contradiction. $\square$

Theorem 6.2. *Suppose $\mathcal{Q} \in \mathcal{Q}(n, d, m)$, h is a closed path in $\mathcal{Q}$, and $h \not\equiv 0$. Then $\deg(h) \leq 2m(d - n) + m$.*

Proof. Using the notation of Lemma 2.9 we have $\deg h \leq m(r + 2t)$ and $r + t \leq d - n + 1$. Lemma 6.1 implies

$$r + 2t \leq 2r - 1 + 2t \leq 2(d - n) + 1$$

and we obtain the required upper bound. $\square$

7 Examples

Lemma 7.1. *Suppose $\mathcal{Q}(n,d,m) \neq \emptyset$. Then there is a $\mathcal{Q} \in \mathcal{Q}(n,d,m)$ and a closed path h in $\mathcal{Q}$ such that $h \not\equiv 0$ and*

1) $\deg(h) \geq M(n,d,m) - m$, *if* $\mathrm{char}(K) = 2$*;*

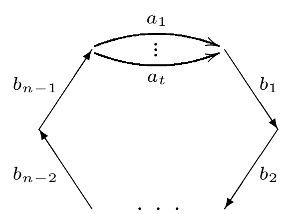
2) $\deg(h) = M(n,d,m)$, *if* $\mathrm{char}(K) \neq 2$, $d \geq n + 2\left[\frac{n-1}{m}\right] + m$ *or* $n = m$*;*

where the definition of $M(n,d,m)$ was given in Section 1.

Proof. Suppose $\mathrm{char}(K) = 2$.

a) If $m = 1$, then $n = 1$. For the quiver $\mathcal{Q}$ with one vertex v and loops $a_1, \ldots, a_d$ in v we have $h = a_1 \cdots a_d \not\equiv 0$ and $\deg(h) = d$.

b) If $m \geq 2$ and $n = m$, then we consider the quiver $\mathcal{Q} \in \mathcal{Q}(n,d,m)$:



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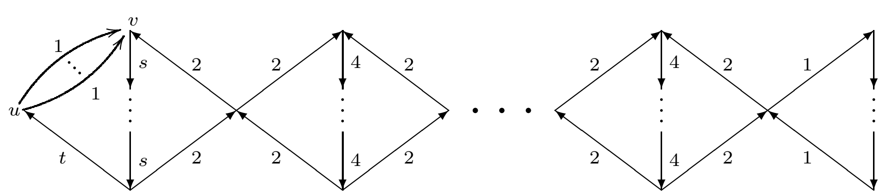
where $t = d - n + 1 \geq 1$. For $h = a_1 b \cdots a_t b$, where $b = b_1 \cdots b_{n-1}$, we have $\deg(h) = tn$ and $h \not\equiv 0$.

c) We assume that $d \geq n + 2\left[\frac{n-1}{m}\right]$ and $n > m \geq 2$. Then $n - 1 = lm + r$ for $l = \left[\frac{n-1}{m}\right] \geq 1$ and $0 \leq r \leq m - 1$. Consider the quiver $\mathcal{Q} \in \mathcal{Q}(n,d,m)$:



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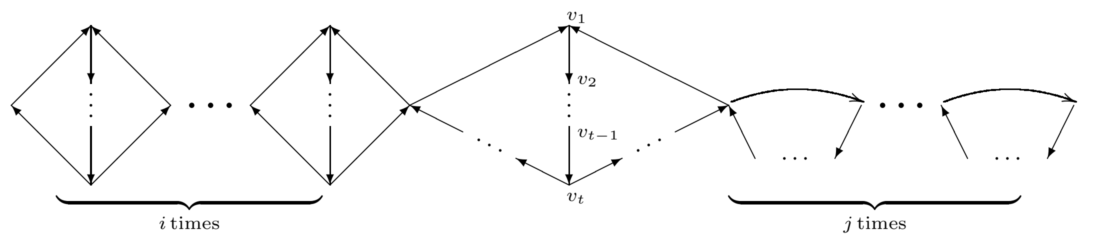
where there are $t = d - n - 2l + 1 \geq 1$ arrows from u to v, the right primitive closed path contains $r + 1$ arrows, any other primitive closed path contains m arrows, and $s = t + 2$. Define $\underline{\delta} \in \Omega_1(\mathcal{Q})$ in such a way that if a number k is assigned to an arrow $a \in \mathrm{arr}(\mathcal{Q})$, then $\delta_a = k$. Since $\underline{\delta} \in \Omega_2(\mathcal{Q})$, there is a closed path h in $\mathcal{Q}$ with $\mathrm{mdeg}(h) = \underline{\delta}$ and $h \not\equiv 0$ by Remark 3.2. It is not difficult to see that $\deg(h) = |\underline{\delta}| = m(d - n - 1) + 2n - (r + 1)$.

d) We assume that $d < n + 2\left[\frac{n-1}{m}\right]$ and $n > m \geq 2$. As above, we have $n - 1 = lm + r$ for

$l \geq 1$ and $0 \leq r \leq m-1$. Consider the quiver $\mathcal{Q} \in \mathcal{Q}(n,d,m)$:



where every primitive closed path contains m arrows, $i, j \geq 0$, $1 \leq t < m$, and

$$\begin{array}{rcl} n & = & m(i+j+2) - j - t, \\ d & = & m(i+j+2) + 2i - t + 1. \end{array}$$

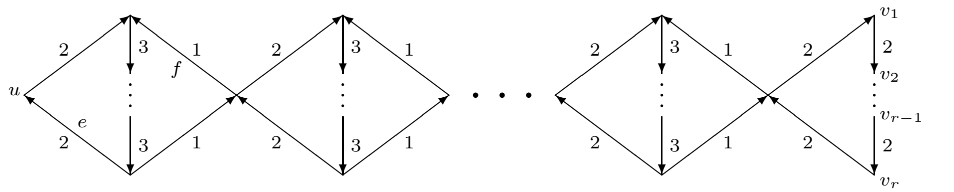
It is not difficult to see that there exist i, j, t satisfying the given conditions. We define $\underline{\delta} \in \Omega_2(\mathcal{Q})$ in a similar way as in part c). Hence $|\underline{\delta}| = 2m(2i+j+1)$ and $M(n,d,m) - |\underline{\delta}| = m$.

e) Suppose $\mathrm{char}(K) \neq 2$ and the condition from part 2) of the lemma holds.

If $m = 1$, then we construct the required h similarly to part a).

If $n = m \geq 2$, then we consider the quiver from part b). We set $h = a_1 b a_1 b$ if $d \in \{n, n+1\}$ and $h = a_1 b a_2 b a_3 b$ if $d > n+1$. Obviously, $\deg(h) = M(n,d,m)$ and $h \not\equiv 0$.

Let $n > m \geq 2$. We define l and r in the same way as in part c) and consider the quiver $\mathcal{Q} \in \mathcal{Q}(n,d,m)$:



Here we assume that we have not depicted some loops in $\mathcal{Q}$. Namely, for $s = d - n - 2l - r - 1 \geq 0$ there are loops a, $b_1, \ldots, b_s$ in the vertex u and loops $c_1, \ldots, c_r$ in vertices $v_1, \ldots, v_r$, respectively. We assign number 1 to loops a, $c_1, \ldots, c_r$ and number 0 to $b_1, \ldots, b_s$. Define $\underline{\delta} \in \Omega_1(\mathcal{Q})$ in such a way that if a number k is assigned to an arrow $x \in \mathrm{arr}(\mathcal{Q})$, then $\delta_x = k$. Let h be a closed path in $\mathcal{Q}$ with $\mathrm{mdeg}(h) = \underline{\delta}$. Since $\deg_w(h) = 3$ for all $w \in \mathrm{ver}(\mathcal{Q})$, we have $\deg(h) = 3n$. Lemma 7.2 (see below) completes the proof. $\square$

Given a closed path $a = a_1 \cdots a_s$ in $\mathcal{Q}$, where $a_i \in \mathrm{arr}(\mathcal{Q})$, we write $\mathrm{tr}(X_a)$ for $\mathrm{tr}(X_{a_s} \cdots X_{a_1})$.

Lemma 7.2. *Using notation from part e) of the proof of Lemma 7.1, we have $h \not\equiv 0$.*

Proof. Since the construction of $\mathcal{Q}$ and h depend on l, we write $\mathcal{Q}_l$ for $\mathcal{Q}$ and h_l for h ($l \geq 1$).

Assume that $h_l \equiv 0$. By Lemma 1.4, $\mathrm{tr}(X_{h_l}) \equiv 0$. Denote $I = \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right)$ and $J = \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right)$. We set $X_a = I$, $X_e = J$, and $X_g = E$ for every arrow $g \notin \{a, e, f\}$

from the left rhombus of $\mathcal{Q}_l$. Since $\mathrm{tr}(I) = \mathrm{tr}(J) = \mathrm{tr}(IJ) = 0$, it is not difficult to see that $\mathrm{tr}(X_{h_{l-1}}) \equiv 0$ in $I(\mathcal{Q}_{l-1}, (2, \ldots, 2))$, where h_0 is defined below. Repeating this procedure, we obtain that $\mathrm{tr}(X_{h_0}) \equiv 0$ in $I(\mathcal{Q}_0, (2, \ldots, 2))$ for

$$h_0 = x_1 y_1 \cdots x_{r+1} y_{r+1} \cdot x_1 \cdots x_{r+1},$$

where $x_1, \ldots, x_{r+1} \in \mathrm{arr}(\mathcal{Q}_0)$, $x_1 \cdots x_{r+1}$ is a closed primitive path in $\mathcal{Q}_0$, y_i is a loop in x_i' $(1 \leq i \leq r+1)$. For $j = 1, 2$ we denote

$$z_{ij} = \left\{ \begin{array}{cl} y_i, & \text{if } j = 1 \\ 1_{x_i'}, & \text{otherwise} \end{array} \right. .$$

Since for all $\pi_1, \ldots, \pi_{r+1} \in \mathcal{S}_2$

$$x_1 z_{1,\pi_1(1)} \cdots x_{r+1} z_{r+1,\pi_{r+1}(1)} \cdot x_1 z_{1,\pi_1(2)} \cdots x_{r+1} z_{r+1,\pi_{r+1}(2)} \equiv \mathrm{sgn}(\pi_1) \cdots \mathrm{sgn}(\pi_{r+1}) h_0,$$

we obtain that $h_0 \not\equiv 0$. Lemma 1.4 implies a contradiction. $\square$

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